The Double Pendulum

Experiment

1. Objectives

Fig. 1: The Double Pendulum experiment

The Double pendulum is in some sense similar to the Furuta pendulum, but slightly more difficult to control. Both experiments together are ideally suited to teach advanced methods of control theory. The main difference is that the transfer function for the inverted double pendulum has two positive poles whereas the inverted Furuta pendulum has only one. This causes that inaccuracies in the mathematical model of the plant are of greater influence for the double pendulum than for the Furuta pendulum. If both experiments are used within one lab, it would be recommendable to start with the Furuta pendulum.

As for the Furuta pendulum, state space methods are applied to design controllers. In the simplest case, the full state is fed back and the controller is designed by pole placement. The students learn to place the poles suitably and to understand the role of the time constant $T_0$ of the differentiators. Even if such a controller is carefully designed, the closed-loop system may be unstable because there is unmodeled high-frequency dynamics in the plant model. By this way, the students learn that with the real plant, the closed-loop system may behave very different to the simulated plant. These dynamics can be identified and a simple redesign yields a stabilizing controller. A systematic way to handle such a situation will be given in the Uncertain Double Pendulum Workshop.

Fig. 2: Stabilization of the inverted pendulum

A slightly more advanced design technique yields the linear quadratic regulator. Additionally, the differentiators can be replaced by an observer such as a Luenberger observer or the famous Kalman filter. A LQ controller in combination with a Kalman filter yields a LQG controller (or $H_2$ controller). The characteristic property of all these methods is that appropriate weights have to be chosen. The students learn how to do this and can see how the choice influences the performance of the controlled double pendulum. They are also asked to use classical concepts like the Nyquist curve. The LQG controller is less sensitive against the unmodeled high-frequency dynamics, has a better performance, and the students learn why.

2. Plant and Simulation Model

The control variable is the input voltage coming from the computer and the output variables are the angle $\delta$ which make the two rods with each other and the deflection angle $\phi$ of the rod which is mounted on the motor shaft. If friction is neglected, the transfer functions for the hanging pendulum has a pole pair on the imaginary axis:

$$G_{\delta_0}(s) = \frac{s^2 + \omega^2_{z1}}{(s^2 + \omega^2_1)(s^2 + \omega^2_2)}$$
For the inverted pendulum, the transfer functions are

\[ G_{\theta_0}(s) = k_2 \frac{s^2 + \omega_2^2}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)} \].

For the inverted pendulum, the transfer functions are

\[ G_{\theta_0}(s) = k_1 \frac{(s - \omega_1)(s + \omega_1)}{(s - \omega_1)(s + \omega_1)(s - \omega_2)(s + \omega_2)} \]

\[ G_{\phi_0}(s) = k_2 \frac{(s - \omega_2)(s + \omega_2)}{(s - \omega_1)(s + \omega_1)(s - \omega_2)(s + \omega_2)} \].

These transfer functions are used for controller design. The full simulation model contains the nonlinearity, viscous and Coulomb friction, limitations and so on. Figure 3 shows the SIMULINK model of the closed-loop system. It may be used for non real-time simulation as well as for real-time simulation. The controller block contains two identical controllers: One serves for the simulation and the other is connected to hardware.

Some of the experiments dealing with the plant are:

- Step responses for small input signals to illustrate the dynamics of the transfer functions
- Step responses for large input signals to compare the linearized model with the nonlinear model
- Step responses to show the effect of Coulomb friction on the dynamics
- Step responses for a short time interval to identify unmodeled high-frequency dynamics
- Responses to sinusoidal inputs to demonstrate the practical application of Bode plots
- Responses to sinusoidal inputs of a high frequency to identify unmodeled high-frequency dynamics.

The identified high-frequency dynamics (IdHFDyn) will be incorporated into the simulation model.

3. Controllers

Various controllers have to be designed and tested. The typical experiments for the closed-loop systems are:

- Generation of a step response for a (constant) command
- Tracking of a non constant reference variable
- Rejection of a disturbance.

Pole placement controller for the hanging pendulum. A controller which stabilizes the hanging pendulum can be designed by pole placement. This controller feeds back the complete state. The rates are obtained by differentiators. The desired poles of
the closed-loop system are chosen such that the Nyquist curve avoids a unit circle centered at −1 and such that the time constants of the differentiators only moderately influence the Nyquist curve and the prescribed poles of the closed-loop system. This controller works very well in the simulation, but fails when applied to the hardware.

The explanation of this surprising behavior is the IdHFDyn. If it is added in the simulation, the closed-loop system becomes unstable in the simulation, too. The parameters of the pole placement controller can be adjusted experimentally to get a satisfactory result.

Pole placement controller for the other equilibrium points. The double pendulum has four equilibrium points, namely

- do-do: lower and upper pendulum hanging,
- up-do: lower pendulum inverted, upper pendulum hanging,
- do-up: lower pendulum hanging, upper pendulum inverted,
- up-up: lower and upper pendulum inverted.

For the remaining three equilibrium points, also pole placement controllers will be designed.

Design of an estimator. In another part of the experiment, observers are designed to estimate the rates. These observers are

- a Luenberger observer
- a Kalman filter.

The accuracy of these observers are tested experimentally for the open-loop system.

Design of \( H_\infty \) controllers. Such a controller will be designed for all four equilibrium points. The design procedure consists of two steps:

- Design of a LQ controller
- Design of a Kalman filter.

The students learn to choose suitable weights and have the possibility to try out their role experimentally.

It can be observed that for the inverted pendulum a limit cycle occurs. The reason for this limit cycle is the Coulomb friction in the bearings. This can be shown by switching on and off the Coulomb friction in the simulation model.

Design of a controller with only one measurement. Such a controller can be obtained by putting together a LQ controller and a Kalman filter.

4. Analysis Tools

In order to get a deeper understanding of controller design and the fundamental properties of the plants and the closed-loop systems, the PendCon SW includes m.files in which the required analysis is prepared. For the plants, the following plots can be generated:

- Pole-zero maps of the plant (in dependency of the linearization point)
- Bode plots of the plant (without and with the high-frequency dynamics).

For controller synthesis and analysis, it is possible to obtain the following plots:

- A plot of the Nyquist curve (without and with the high-frequency dynamics)
- Bode plot of the loop transfer function
- Bode plots of the other transfer functions of the closed-loop system. By this way, it is possible to determine the bandwidth of the closed-loop system, to study disturbance rejection properties of the CLS or to analyze the control effort. Additionally, robustness properties of the closed-loop system can be judged from these curves.
- Pole-zero maps of the CLS.

By this way, the students have the possibility to learn how the design parameters for the various controllers effect the poles of the CLS and the Bode plots of the transfer function of the CLS (this is necessary for \( H_\infty \) synthesis). Besides this, they learn why the IdHFDyn can destabilize the closed-loop system.

5. Simulation Results

The following figures show some simulation results which are obtained from the experiments. In these figures, the red (blue) lines represent the measured (simulated) quantities, respectively. Figure 4 shows a pulse response for the hanging pendulum. It can be seen that the simulation model is very accurate. In Fig. 5 tracking of a command with a LQG controller for the inverted pendulum is shown.

Video Clips. We finally mention the video clips belonging to the experiments which can be found on www.pendcon.de. The reader is encouraged to visit this homepage where various clips for the double pendulum are presented.
Fig. 4: Response to a pulse

Fig. 5: Tracking of a command for the inverted pendulum